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Adams operations, iterated integrals and free loop spaces cohomology

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Abstract

We construct by geometrical means a weight-decomposition of Chen's iterated integrals in the case of a free loop fibration pullback.

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0. Introduction

Weight decompositions for Hochschild and cyclic homology of (eventually graded and differential) commutative algebras were defined and studied independently and simultaneously by various authors [3, 8, 10, 12].

[3] was concerned with the Quillen–Sullivan rational homotopy theory, whereas [8, 10, 12] concentrated on homological methods involving the combinatorics of symmetric groups. Means were different, and so were the definitions of weight decompositions; it was anyway proved later on in [14] that those combinatorial and topological definitions agree.

Recently, [1] and [2] showed that Adams operations, which are the operations providing the weight decomposition, do have a geometric interpretation as far as commutative differential graded algebras (CDGAs for short) are concerned. Indeed, let X be a smooth 1-connected manifold and $\Omega^*(X)$ its real or complex de Rham algebra; then the Hochschild homology of the CDGA $\Omega^*(X)$ identifies with the singular cohomology of the free loop space of X:

$$HH_{-*}(\Omega^*(X),d) \cong H^*(X^{S^1};k);$$

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k being the field of real or complex numbers, according to the choice of the ground field for differential forms. The key result of [1] is the identification of Adams operations on $HH_{-*}(\Omega^*(X),d)$ (resp. on the corresponding cyclic homology groups) with the operations on $H^*(X^{S^1},k)$ (resp. on the corresponding equivariant cohomology groups) induced by iteration of loops.

We are concerned further on with the study of pullbacks of the free loop fibration in the setting of Chen's iterated integrals theory, and recover by those means the results of [1] on free loop spaces. It should be pointed out, that our methods are entirely different from those previously used in the subject. We go back indeed to the original philosophy of Chen, which requires the process of integrating differential forms over simplices, as emphasized in [4]. This allows us to make use of the geometrical results of [13].

Our main result reads: let X be a 1-connected differentiable manifold, whose cohomology is of finite type, let $Y \xrightarrow{i} X$ be a differentiable map and $X_Y^{S^1} \longrightarrow Y$ be the pullback of the free loop fibration along i:

$$\begin{array}{ccc} X_Y^{S^1} & \longrightarrow & X^{S^1} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{i} & X; \end{array}$$

then the complex of iterated integrals which computes the cohomology of $X_Y^{S^1}$ has a canonical decomposition – in fact, a weight decomposition. The basic geometrical remark is that, at the level of iterated integrals, to iterate loops is nothing but a geometrical operation on the simplices over which differential forms have to be integrated.

As Micheline Vigué pointed out to me, the computations in [3] can be generalized and performed on a Sullivan model for $X_Y^{S^1}$. We conclude showing that the so-obtained decomposition of a Sullivan model for $X_Y^{S^1}$ coincides with the one we previously obtained at the level of iterated integrals.

Notations. we shall write $\boldsymbol{N}$ (resp. $\boldsymbol{Q}$, $\boldsymbol{R}$, etc.) for the set of entire (resp. rational, real, etc.) numbers.

The end of a proof is indicated by an "open square" $\square$.

1. Iterated integrals

We just recall those results of [7] we shall need later on.

1.1. Chen complexes

Let X be a 1-connected differentiable manifold whose cohomology is of finite type, and $Y \xrightarrow{i} X$ a differentiable map. Let $\Omega^*(Y)$ be, as usual, the de Rham complex of C^∞ real or complex differentiable forms on Y and $\Omega^*(X)(-1)$ the complex whose degree p elements are $p+1$-differentiable forms on X.

By definition, *the Chen complex* $Ch^*(X,Y)$ relative to the map i is the cochain complex whose elements are the elements of:

$$\bigoplus_{k\in N} \Omega^*(Y)\otimes(\Omega^*(X)(-1))^{\otimes k}.$$

We write $u\otimes[\omega_1|\dots|\omega_k]$ for $u\otimes\omega_1\otimes\cdots\otimes\omega_k$ when u (resp. each ω_i) is a homogeneous element of $\Omega^*(Y)$ (resp. of $\Omega^*(X)$).

If α is a degree k element of a graded vector space, let us define $J(\alpha)$ as

$$J(\alpha) := (-1)^k\cdot\alpha.$$

Observe that, if $\forall i\in[1,k],\ w_i\in\Omega^{n_i}(X)$, then:

$$J[w_1|\dots|w_k] = (-1)^{\sum_{i=1}^k(n_i-1)}[w_1|\dots|w_k] = (-1)^k[Jw_1|\dots|Jw_k].$$

If $\omega\in\Omega^n(X)$, set:

$$\omega^* := \Omega^*(i)(\omega).$$

Let d be the usual differentiation map of differential forms. The coboundary map d_C in $Ch^*(X,Y)$ is given by [7, formula (4.2.2)],

$$\begin{aligned} d_C(u\otimes[w_1|\dots|w_k]) := {} & du\otimes[w_1|\dots|w_k] - Ju\wedge w_1^*\otimes[w_2|\dots|w_k]\\ & + \sum_{1\le i\le k}(-1)^i Ju\otimes[Jw_1|\dots|Jw_{i-1}|dw_i|w_{i+1}|\dots|w_k]\\ & - \sum_{1\le i<k}(-1)^i Ju\otimes[Jw_1|\dots|Jw_{i-1}|Jw_i\wedge w_{i+1}|w_{i+2}|\dots|w_k]\\ & +(-1)^{(deg\, w_k-1)\left(\sum_{i=1}^{k-1}(deg\, w_i-1)\right)} Ju\wedge w_k^*\otimes[w_1|\dots|w_{k-1}]. \end{aligned}$$

It should be pointed out that the Chen complex has nontrivial negative degree components (choose $u\in\Omega^n(Y)$ and $\omega_1,\dots,\omega_k$ in $\Omega^0(X)$ s.t. $k>n$). However, the theory could be worked out with a cochain complex having only positive degree components [6].

1.2. Plots and forms

A key step when defining iterated integrals is the construction of a suitable "de Rham complex" for path spaces. For our purposes, it is enough to make this construction for the spaces $X_Y^{S^1}$ – see [7] for generalizations.

Further on, we identify the unit circle S^1 with $\boldsymbol{R}/\boldsymbol{Z}$. By definition an element of X^{S^1}, or a *loop* on X, is a C^∞ map from S^1 to X.

Recall that $X_Y^{S^1}$ is defined via the pullback diagram:

$$\begin{tikzcd} X_Y^{S^1} \arrow[r, "i_\pi"] \arrow[d, "\pi_i"] & X^{S^1} \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X, \end{tikzcd}$$

where π is the free loop space fibration, that is the map which sends a loop $\gamma \in X^{S^1}$ to its origin $\gamma(0)$.

Let A be a convex open subset of $\boldsymbol{R}^n$. If α is a map from A to $X^{S^1} = Hom_{C^\infty}(S^1, X)$, define ϕ_α to be the induced map from $A \times S^1$ to X:

$$\phi_\alpha(a,t) := \alpha(a)(t).$$

Definition 1.1. A map $\alpha : A \longmapsto X_Y^{S^1}$ is a plot if the following conditions are satisfied:
(i) $\pi_i \circ \alpha$ is a C^∞ map from A to Y.
(ii) $\phi_{i_\pi \circ \alpha}$ is a C^∞ map from $A \times S^1$ to X.

Define Θ to be the category whose objects are euclidean convex sets and whose arrows are the C^∞ maps. If A (resp. $f : A \longrightarrow B$) is an object (resp. an arrow) of Θ, define $F(A)$ to be the set of all plots from A to $X_Y^{S^1}$ (resp. the set morphism:

$$F(f) : F(B) \longrightarrow F(A)$$
$$F(f)(\beta) := \beta \circ f, \quad \forall \beta \in F(B)).$$

Clearly, F is a contravariant set functor.

Recall Ω^p (resp. Ω^*) is the usual contravariant functor from differentiable spaces to vector spaces (resp. to CDGAs) and write Ω_s^p (resp. Ω_s^*) for the corresponding set functor, that is Ω^p (resp. Ω^*) viewed as a functor from spaces to sets.

Definition 1.2. A p-form (resp. a form) on $X_Y^{S^1}$ is any natural transformation from F to Ω_s^p (resp. Ω_s^*).

Let now $\wedge^p(X_Y^{S^1})$ (resp. $\wedge^*(X_Y^{S^1})$) be the set of p-forms (resp. forms) on $X_Y^{S^1}$. Since Ω^p (resp. Ω^*) is a vector space (resp. a CDGA) functor, $\wedge^p(X_Y^{S^1})$ (resp. $\wedge^*(X_Y^{S^1})$) is canonically provided with a structure of vector space (resp. CDGA). According to Chen [7, Sections 1.2 and 1.5], we call $\wedge^*(X_Y^{S^1})$, the *de Rham complex of* $X_Y^{S^1}$.

1.3. Iterated integrals

The most important step in iterated integrals calculus is the construction of a map from the Chen complex $Ch^*(X, Y)$ to the de Rham complex $\wedge^*(X_Y^{S^1})$.

Recall S^1 and $\boldsymbol{R}/\boldsymbol{Z}$ have been identified. Let α be a plot from A to $X_Y^{S^1}$, and define μ_α to be the following (1-periodic in the $\boldsymbol{R}$-direction) map:

$$\mu_\alpha : A \times \boldsymbol{R} \longrightarrow X,$$

$$\mu_\alpha(a,t) := \phi_{i_\pi \circ \alpha}(a,t).$$

If $\omega \in \Omega^{p+1}(X)$, define $\omega_{\mu_\alpha} \in \Omega^{p+1}(A \times \boldsymbol{R})$ as

$$\omega_{\mu_\alpha} := \Omega^*(\mu_\alpha)(\omega).$$

Let dt be the standard generator of the $\Omega^0(\boldsymbol{R})$-module $\Omega^1(\boldsymbol{R})$ and recall a $p+1$ form μ on $A\times\boldsymbol{R}$ has a unique expression $dt\wedge\mu'+\mu''$, where μ' and μ'' belong to the subalgebra of $\Omega^*(A \times \boldsymbol{R})$ generated by $\Omega^*(A)$ and $\Omega^0(A \times \boldsymbol{R})$. Then, define $\omega_\alpha \in \Omega^p(A \times \boldsymbol{R})$ to be

$$\omega_\alpha := {\omega_{\mu_\alpha}}',$$

and remark ω_α can be expressed as a sum of products of C^∞ functions on $A \times \boldsymbol{R}$ by p-differential forms on A.

More generally, let ρ_α^k be the morphism:

$$\rho_\alpha^k : A \times \boldsymbol{R}^k \longrightarrow Y \times X^k$$

$$\rho_\alpha^k(a,t_1,\dots,t_k) := (\pi_i \circ \alpha(a), \mu_\alpha(a,t_1),\dots,\mu_\alpha(a,t_k)).$$

Let pr_j (resp. pr_0) be the projection map from $A \times \boldsymbol{R}^k$ to $A \times \boldsymbol{R}$ (resp. A):

$$pr_j(a,t_1,\dots,t_k) := (a,t_j),$$

(resp. $pr_0(a,t_1,\dots,t_k) := a$). If

$$\tilde{\omega} = u \otimes [\omega_1|\dots|\omega_k] \in \Omega^p(Y) \otimes \Omega^{p_1+1}(X) \otimes \dots \otimes \Omega^{p_k+1}(X),$$

define $\bar{u}$, $\bar{\omega}_j$ and $\tilde{\omega}_\alpha$, respectively, by

$$\bar{u} := \Omega^*(\pi_i \circ \alpha \circ pr_0)(u) \in \Omega^p(A \times \boldsymbol{R}^k),$$

$$\bar{\omega}_j := \Omega^*(pr_j)(\omega_j)_\alpha \in \Omega^{p_j}(A \times \boldsymbol{R}^k),$$

$$\tilde{\omega}_\alpha := \bar{u} \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k \in \Omega^{p+p_1+\cdots+p_k}(A \times \boldsymbol{R}^k).$$

It should be pointed out that $\tilde{\omega}_\alpha$ belongs to the subalgebra of $\Omega^*(A \times \boldsymbol{R}^k)$ generated by $\Omega^*(A)$ and $\Omega^0(A \times \boldsymbol{R}^k)$.

We are now in position to define the announced map from $Ch^*(X,Y)$ to $\wedge^*(X_Y^{S^1})$. Let Δ_1^n be the n-simplex:

$$\Delta_1^n := \{(t_1,\dots,t_n) \in [0,1]^n | t_1 \leq \cdots \leq t_n\}.$$

We associate to each $\tilde{\omega} \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$ a map $\int_{\tilde{\omega}}$ from plots $\alpha : A \longrightarrow X_Y^{S^1}$ to differential forms on A by

$$\int_{\tilde{\omega}} \alpha := \int_{\Delta_1^n} \tilde{\omega}_\alpha.$$

Each morphism $\int_{\tilde{\omega}}$ is easily seen to define a natural transformation from F to Ω_s^*, that is, a form on $X_Y^{S^1}$. By definition the form $\int_{\tilde{\omega}}$ on $X_Y^{S^1}$ is the *iterated integral* associated to $\tilde{\omega}$ (see [5, 7, 4]).

Let $It^*(X, Y)$ be the set of all iterated integrals:

$$It^*(X, Y) := \left\{ \int_{\tilde{\omega}}, \tilde{\omega} \in Ch^*(X, Y) \right\}.$$

By definition, $It^*(X, Y) \subset \wedge^*(X_Y^{S^1})$.

Theorem 1.3 (Chen [7, Theorems 4.2.1 and 4.3.1]). *The set of iterated integrals $It^*(X, Y)$ is a subcomplex of $\wedge^*(X_Y^{S^1})$ whose cohomology is isomorphic to the singular cohomology of $X_Y^{S^1}$. Moreover, the map $\tilde{\omega} \longrightarrow \int_{\tilde{\omega}}$ from $Ch^*(X, Y)$ to $It^*(X, Y)$ is a quasi-isomorphism.*

2. Weight decompositions

2.1. Generalized iterated integrals

Let P be any convex polytope of $\mathbf{R}^n$, that is the convex envelope of a finite set of points. We can repeat the process of Section 1.3 and define for each $\tilde{\omega} \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$ a natural transformation $\int_{P,\tilde{\omega}}$ from F to Ω_s^*, that is, a form on $X_Y^{S^1}$, by

$$\forall A \in Obj(\Theta),\ \forall \alpha \in F(A),\ \int_{P,\tilde{\omega}} \alpha := \int_P \tilde{\omega}_\alpha.$$

We call $\int_{P,\tilde{\omega}}$ a *generalized iterated integral* and write $\int_P$ the map from $\Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$ to $\wedge^*(X_Y^{S^1})$: $\int_P \tilde{\omega} := \int_{P,\tilde{\omega}}$.

Lemma 2.1. *Suppose $P = P_1 \cup P_2$, P_1 and P_2 are convex polytopes of $\mathbf{R}^n$ and $P_1 \cap P_2$ is degenerated – that is, $P_1 \cap P_2$ is included in some hyperplane of $\mathbf{R}^n$. Then*:

$$\int_P = \int_{P_1} + \int_{P_2}.$$

Proof. The lemma follows immediately from the definition of generalized iterated integrals. □

Lemma 2.2. *Let $t \in \boldsymbol{Z}^n$ and let $P_t := P + t$, that is, let P_t be the convex polytope deduced from P by translation by t. Then*:

$$\int_{P_t} = \int_P .$$

Proof. The lemma follows from the periodicity of the map μ_α. □

If $\sigma \in S_n$, define now Δ^n_σ to be

$$\Delta^n_\sigma := \{(t_{\sigma(n)}, \dots, t_{\sigma(1)}) \in [0,1]^n | t_n \leq \cdots \leq t_1\},$$

and set: $\int_\sigma := \int_{\Delta^n_\sigma}$.

Let Ψ^k be the dilation by k in $\boldsymbol{R}^n$. We have

$$\Psi^k(\Delta^n_1) = \{(t_1, \dots, t_n) \in \boldsymbol{R}^n | 0 \leq t_1 \leq \cdots \leq t_n \leq k\}.$$

Recall from [13] that $\Psi^k(\Delta^n_1)$ decomposes as a union of simplices $\Delta^n_\sigma + t$, $t \in \boldsymbol{N}^n$:

$$\Psi^k(\Delta^n_1) = \bigcup_{\sigma \in S_n} \left(\bigcup_{t \in S_k(\sigma)} \Delta^n_\sigma + t \right),$$

where for each σ, $S_k(\sigma)$ is a finite subset of $\boldsymbol{N}^n$, so that by Lemmas 2.1 and 2.2 there exist coefficients c^k_σ with

$$\int_{\Psi^k(\Delta^n_1)} = \sum_{\sigma \in S_n} c^k_\sigma \int_\sigma ;$$

the coefficients c^k_σ being those which appear in [13], Section 2.1.

Corollary 2.3. *The map $\int_{\Psi^k(\Delta^n_1)}$ from $\Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$ to forms decomposes*:

$$\int_{\Psi^k(\Delta^n_1)} = \sum_{\sigma \in S_n} c^k_\sigma \cdot \int_\sigma , \quad c^k_\sigma \in \boldsymbol{N}.$$

In fact, it follows from [13], that this decomposition is polynomial in k.

Proposition 2.4. *There exist rational coefficients a^i_σ, $\sigma \in S_n$, $i \in [1,n]$ with*

$$\int_{\Psi^k(\Delta^n_1)} = \sum_{i \in [1,n]} \left[\sum_{\sigma \in S_n} a^i_\sigma \cdot \int_\sigma \right] \cdot k^i .$$

Proof. See [13], Section 2.2. □

2.2. Iteration of loops

Let us return to loops and define Φ^k to be the endomorphism of $X_Y^{S^1}$ induced by k-fold iteration of loops. Precisely, let $\bar{\Phi}^k$ be the endomorphim of X^{S^1}:

$$\forall t \in \boldsymbol{R}/\boldsymbol{Z},\ \forall \zeta \in Hom_{C^\infty}(S^1,X), \bar{\Phi}^k(\zeta)(t) := \zeta(k \cdot t).$$

Then, Φ^k is defined to be the pullback along $i : Y \longrightarrow X$ of the endomorphism $\bar{\Phi}^k$ of the free loop space fibration $X^{S^1} \longrightarrow X$.

The endomorphisms Φ^k of $X_Y^{S^1}$ induce endomorphisms Φ^{k^*} of $\wedge^*(X_Y^{S^1})$, the de Rham complex of $X_Y^{S^1}$. In fact, Φ^{k^*} is also an endomorphism of $It^*(X,Y)$ and we are going to compute its action on this cochain complex. By Theorem 1.3 this implies that we shall also be able to compute the induced action on $H^*_{sing}(X_Y^{S^1})$.

Lemma 2.5. *Let $v \in S_n$ be the "permutation of maximal lenght" in S_n, that is,*

$$\forall i \in [1,n] \quad v(i) := n+1-i.$$

Let $\sigma \in S_n$ and $\tilde{\omega} = u \otimes [\omega_1|\ldots|\omega_n] \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$, then

$$\int_\sigma \tilde{\omega} = \int_{\sigma(\tilde{\omega})},$$

where, by definition:

$$\sigma(\tilde{\omega}) := \varepsilon(\sigma,\tilde{\omega}) \cdot u \otimes [\omega_{v\cdot\sigma^{-1}\cdot v(1)}|\cdots|\omega_{v\cdot\sigma^{-1}\cdot v(n)}].$$

Proof. Check the formula in the definition of generalized iterated integrals; $\varepsilon(\sigma,\tilde{\omega}) = \pm 1$ is a "graded signature" depending on σ; $deg\,\omega_1,\ldots,deg\,\omega_n$, whose definition is tedious but straightforward. $\square$

Lemma 2.6. $\forall \tilde{\omega} \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}$:

$$\Phi^{k^*}\left(\int_{\tilde{\omega}}\right) = \int_{\Psi^k(\Delta_1^n),\tilde{\omega}}.$$

Proof. Indeed, let $\alpha : A \longrightarrow X_Y^{S^1}$ be a plot and set $\Phi^k(\alpha) := \Phi^k \circ \alpha$. Then:

$$\Phi^{k^*}\left(\int_{\tilde{\omega}}\right)(\alpha) = \int_{\tilde{\omega}} \Phi^k(\alpha) = \int_{\Delta_1^n} \tilde{\omega}_{\Phi^k(\alpha)}.$$

Besides, the morphism $\rho^n_{\Phi^k(\alpha)}$, as defined in Section 1.3, factorizes:

$$\begin{array}{ccccc} A \times \boldsymbol{R}^n & \xrightarrow{1\times k} & A \times \boldsymbol{R}^n & \xrightarrow{\rho^n_\alpha} & Y \times X^n \\ (a,x_1,\ldots,x_n) & \longmapsto & (a,k\cdot x_1,\ldots,k\cdot x_n) & \longmapsto & (\pi_i \circ \alpha(a), \mu_\alpha(a,k\cdot x_1),\ldots,\mu_\alpha(a,k\cdot x_n)), \end{array}$$

so that, by a change of variables in the integral $\int_{\Delta_1^n} \tilde{\omega}_{\Phi^k(\alpha)}$, we get

$$\int\limits_{\Delta_1^n} \tilde{\omega}_{\Phi^k(\alpha)} = \int\limits_{\Psi^k(\Delta_1^n)} \tilde{\omega}_\alpha. \qquad \square$$

Corollary 2.7. *With the a_σ^is as in Proposition 2.4*:

$$\forall \tilde{\omega} \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n} : \ \Phi^{k^*}\left(\int\limits_{\tilde{\omega}}\right) = \sum_{i=1}^n \left[\sum_{\sigma \in S_n} a_\sigma^i \int\limits_{\sigma(\tilde{\omega})}\right] \cdot k^i.$$

Proof. The corollary follows from Lemmas 2.6 and 2.5 and Proposition 2.4. $\square$

Recall from [13] that, if we define $e_n^i \in \mathbf{Q}[S_n]$ by

$$\forall i \in [1,n] \quad e_n^i := \sum_{\sigma \in S_n} a_\sigma^i \cdot \sigma,$$

then $(e_n^i)_{i \in [1,n]}$ is a complete family of orthogonal idempotents in $\mathbf{Q}[S_n]$, that is,

$$e_n^i \circ e_n^j = \delta_i^j \cdot e_n^i$$

and

$$\sum_{i=1}^n e_n^i = 1.$$

Define also $e_n^0 \in \mathbf{Q}[S_n]$ to be 0 if $n \neq 0$ and 1 if $n = 0$; and define endomorphisms ε_i of $It^*(X,Y)$ by

$$\forall \tilde{\omega} = u \otimes [\omega_1 | \ldots | \omega_n] \in \Omega^*(Y) \otimes (\Omega^*(X)(-1))^{\otimes n}, \ \varepsilon_i\left(\int\limits_{\tilde{\omega}}\right) := \int\limits_{e_n^i(\tilde{\omega})}$$

By Corollary 2.7, ε_i could also be defined as a linear combination of Φ^{k^*}s by inversion of a Van der Monde matrix, so that, since the Φ^{k^*} are cochain complex endomorphisms of $It^*(X,Y)$, ε_i is a well-defined cochain complex endomorphism of $It^*(X,Y)$

Moreover, since $(e_n^i)_{i \in [1,n]}$ is a complete family of orthogonal idempotents, the ε_is are mutually orthogonal projectors:

$$\varepsilon_i \circ \varepsilon_j = \delta_i^j \cdot \varepsilon_i,$$

Theorem 2.8. *The complex $It^*(X,Y)$ of iterated integrals has a weight decomposition under the action of the Φ^{k^*}s. That is, for all $k \in \mathbf{N}$ and all $\tilde{\omega}$ as in Corollary 2.7*:

$$\Phi^{k^*}\left(\int\limits_{\tilde{\omega}}\right) = \sum_{i=1}^n k^i \ \varepsilon_i\left(\int\limits_{\tilde{\omega}}\right),$$

and this induces a cochain complex decomposition of $It^*(X,Y)$:

$$It^*(X,Y) = \bigoplus_{i=0}^{\infty} It^{*,i}(X,Y),$$

where

$$It^{*,i}(X,Y) := \varepsilon_i \cdot It^*(X,Y).$$

Proof. The theorem sumarizes the preceding results and remarks. □

Corollary 2.9. *The singular cohomology of* $X_Y^{S^1}$ *has a weight decomposition under the action of operations* Φ^{k^*} *induced from iteration of loops on* X^{S^1}. *Precisely, let* $H^{n,i}(X_Y^{S^1},\kappa)$ *be the eigenspace of* $H^n(X_Y^{S^1},k)$ *associated to the eigenvalue* k^i *of* Φ^{k^*}, $\kappa = \boldsymbol{R}$ *or* $\boldsymbol{C}$, *then*

$$H^n(X_Y^{S^1},\kappa) = \bigoplus_{i=0}^{n} H^{n,i}(X_Y^{S^1},\kappa)$$

and, moreover,

$$H^{n,0}(X_Y^{S^1},\kappa) \cong H^n(Y,\kappa).$$

Proof. By Theorems 1.3 and 2.8 it is clear that $H^n(X_Y^{S^1},\kappa)$ decomposes:

$$H^n(X_Y^{S^1},\kappa) = \bigoplus_{i=0}^{\infty} H^{n,i}(X_Y^{S^1},\kappa).$$

We are reduced to prove that: $\forall i > n \quad H^{n,i}(X_Y^{S^1},\kappa) = 0$.

Let us choose a graded subalgebra A of $\Omega^*(X)$, connected (i.e. $A^0 \cong \kappa$) and satisfying the property:

$$A^1 \cap d\Omega^0(X) = 0,$$

and require also that the inclusion of A in $\Omega^*(X)$ be a quasi-isomorphism.

Since A is connected, it is canonically provided with an augmentation. Let A^+ be the augmentation ideal, $A^+ = \bigoplus_{n\geq 1} A_n$. Then,

$$Ch^*(A,Y) := \Omega^*(Y) \oplus \left(\bigoplus_{n\geq 1} \Omega^*(Y) \otimes (A^+(-1))^{\otimes n} \right)$$

is a subcomplex of $Ch^*(X,Y)$. By [7, Theorem 4.2.1], the process of iterated integration maps isomorphically $Ch^*(A,Y)$ to a subcomplex $It^*(A,Y)$ of $It^*(X,Y)$ and, moreover, by [5, (4.1.2)–(4.1.4)], the inclusion $It^*(A,Y) \longrightarrow It^*(X,Y)$ is an quasi-isomorphism.

By corollary 2.7, $It^*(A,Y)$ is stable under the action of Φ^{k^*}, so that we can define "Adams operations" Ψ^k on $Ch^*(A,Y)$ by

$$\Psi^k(\tilde{\omega}) := \int^{-1} \circ \Phi^{k^*}\left(\int_{\tilde{\omega}}\right),$$

where $\int^{-1}$ is the inverse of the map:

$$\begin{array}{ccc} Ch^*(A,Y) & \xrightarrow{\cong} & It^*(A,Y) \\ \tilde{\omega} & \longmapsto & \int_{\tilde{\omega}} \cdot \end{array}$$

Then, by Theorem 2.8:

$$\forall \tilde{\omega} \in \Omega^*(Y) \otimes (A^+(-1))^{\otimes n},\ \Psi^k(\tilde{\omega}) = \sum_{i=1}^{n} k^i \cdot e_n^i(\tilde{\omega}).$$

Define an increasing filtration of $Ch^*(A,Y)$ whose nth term is

$$Ch_n^*(A,Y) := \Omega^*(Y) \oplus \left(\bigoplus_{1 \le m \le n} \Omega^*(Y) \otimes (A^+(-1))^{\otimes m} \right)$$

and consider the associated spectral sequence. Define $\tilde{H}^*(X,\kappa)$ to be the reduced cohomology of X. Since A is quasi-isomorphic to $\Omega^*(X)$, and since X is 1-connected, we have

$$E_1^{p,q} = 0 \quad \text{if } q < 0,$$

$$E_1^{0,q} = H^q(Y,\kappa),$$

and, in general,

$$E_1^{p,q} = (H^*(Y,\kappa) \otimes (\tilde{H}^*(X,\kappa)(-1))^{\otimes p})_{p+q}.$$

Finally, since the action of the Ψ^ks is compatible with the filtration and with coboundaries, it is enough to prove that for $p \ge 0$, $q \ge 0$, $E_1^{p,q}$ decomposes under the action of Ψ^k, the eigenvalues being a subset of $\{1, k, \dots, k^p\}$. But, by the identity $\Psi^k(\tilde{\omega}) = \sum_{i=1}^n k^i \cdot e_n^i(\tilde{\omega})$, this property is already true at the E_0-level, and the proof is complete. □

3. Rational homotopy

Let us turn now to the rational homotopy point of view.

We omit details and refer the reader to [16, 15, 3, 11, 9] for foundational material.

3.1. Models of free loop fibrations

Let X be a 1-connected topological space whose cohomology is of finite type, and $i: Y \longrightarrow X$ a continuous map.

Recall from [16] that, if $(\wedge Z, d)$ is a (Sullivan) minimal model for X, then a minimal model of X^{S^1} is $(\wedge Z \otimes \wedge \bar{Z}, \bar{D})$ with

$$(\bar{Z})^p = Z^{p+1},\ \bar{D}(z) = dz,\ \bar{D}(\bar{z}) = -s\ dz,$$

where s is the unique derivation satisfying: $s(z) = \bar{z},\ s(\bar{z}) = 0$.

Let $(\wedge Z, d) \longrightarrow (\wedge Z \otimes \wedge V, \delta)$ be a $\wedge$-minimal model in the sense of Halperin [11] of the map $Y \longrightarrow X$. Remark first that

$$(\wedge Z \otimes \wedge V, \delta) \otimes_{(\wedge Z, d)} (\wedge Z \otimes \wedge \bar{Z}, \bar{D}) \cong (\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta}),$$

with: $\bar{\delta}_{|\wedge Z \otimes \wedge V} = \delta$ and $\bar{\delta}(\bar{z}) = \bar{D}(\bar{z})$.

By [11, 20.6], $(\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta})$ is a model for $X_Y^{S^1}$.

Proposition 3.1. *The model $(\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta})$ for $X_Y^{S^1}$ decomposes as a direct sum:*

$$(\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta}) = \bigoplus_p (\wedge Z \otimes \wedge V \otimes \wedge^p \bar{Z}, \bar{\delta})$$

Proof. Indeed, by definition, $\bar{\delta}_{|\bar{Z}}$ is a map from $\bar{Z}$ to $\wedge Z \otimes \wedge V \otimes \bar{Z}$, so that $\bar{\delta}$ preserves $\wedge Z \otimes \wedge V \otimes \wedge^p \bar{Z}$. $\square$

3.2. A comparison result

If, following [1], we define *infinitesimal Adams operations* ϕ^n on $(\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta})$ by requiring that the ϕ^ns be algebra endomorphisms, and that

(i) $\phi^n(\alpha) = \alpha\ ,\quad \forall \alpha \in \wedge Z \otimes \wedge V,$

(ii) $\phi^n(\bar{z}) = n \cdot \bar{z} \quad \forall \bar{z} \in \bar{Z},$

then $\wedge Z \otimes \wedge V \otimes \wedge^k \bar{Z}$ identifies with the eigenspace associated to the eigenvalue n^k of ϕ^n.

Corollary 3.2. *The rational cohomology of $X_Y^{S^1}$ decomposes under the action of infinitesimal Adams operations:*

$$H^n(X_Y^{S^1}, \mathbf{Q}) = \bigoplus_{i=0}^{n} H^{n,i}(X_Y^{S^1}, \mathbf{Q}),$$

where $H^{*,i}(X_Y^{S^1}, \mathbf{Q}) := H^*(\wedge Z \otimes \wedge V \otimes \wedge^i \bar{Z}, \bar{\delta})$.

By [1, Theorem 3.2], if $Y = X$, that is, $V = 0$, the infinitesimal Adams operations ϕ^n on the minimal model $(\wedge Z \otimes \wedge \bar{Z}, \bar{D})$ of X^{S^1} are nothing but rational homotopy models of the iteration of loops maps on X^{S^1} (that is of maps $\bar{\Phi}^n$ of section 2.2), so that in general infinitesimal Adams operations ϕ^n on $(\wedge Z \otimes \wedge V \otimes \wedge \bar{Z}, \bar{\delta})$ are rational homotopy models of the maps Φ^n on $X_Y^{S^1}$ (see Section 2.2).

Proposition 3.3. *Up to a change of coefficient field, decompositions of Corollaries* 2.9 *and* 3.2 *identify.*

The proposition is clear from the preceeding results and remarks since both decompositions are weight-decompositions of cohomology groups under the action of cohomology maps induced by the endomorphisms Φ^n of $X_Y^{S^1}$.

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